

## Boundary Value Problems for Partial Differential Equations with Exponential Dichotomies

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We are extending the notion of exponential dichotomies to partial differential evolution equations on the  $n$ -torus. This allows us to give some simple geometric criteria for the existence of solutions to certain nonlinear Dirichlet boundary value problems. © 1992 Academic Press, Inc.

### 1. INTRODUCTION

The aim of this paper is to provide some simple geometric criteria for the existence of solutions to some nonlinear boundary value problems. We will consider evolution problems of the following kind,

$$\frac{\partial}{\partial t} w = A(t) w + H(t, x, w), \quad (1.1)$$

where  $w$  is a  $C^n$  valued function of  $x$  and  $t$ ,  $x \in T^m$ ,  $t \in [t_0, t_1]$ ,  $H$  is some sufficiently smooth function of its arguments, and  $A$  is a linear differential operator in  $x$  with time dependent coefficients. We will consider Dirichlet boundary conditions at  $\{t_0\} \times T^m$  and  $\{t_1\} \times T^m$  and require the solution to be  $2\pi$ -periodic in each of the  $x_i$ 's.

When  $A$  has degree 0, that is, when  $A$  is just a time dependent matrix,  $x$  enters as a parameter and one is in a finite dimensional setting. Such problems go back to Sil'nikov [Si] and are extensively discussed by Bo Deng [BD] in the time independent case. (Our main theorem can easily be seen to solve the finite dimensional time dependent Sil'nikov problem.) The basic idea we will be using goes back to Sil'nikov. The

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following simple example illustrates the essential geometry: Consider a real two dimensional system

$$\dot{w} = Aw, \quad w(t_0) \in M_0, w(t_1) \in M_1. \quad (1.2)$$

Suppose that  $A$  is hyperbolic, i.e., the eigenvalues have non-zero real part, one positive, one negative. We will denote the stable manifold of  $A$  by  $W^s$  and the unstable manifold by  $W^u$ . The phase portrait for (1.2) is given in Fig. 1.1. Hence, if  $M_0$  is a line transverse to  $W^s$  and  $M_1$  is a line transverse to  $W^u$ , it is easy to see that there is a unique solution of (1.2). The idea is to follow a disk in  $M_0$  under the forward flow of (1.2). One then uses the hyperbolicity of  $A$  to estimate how close to  $W^u$  it will lie. Doing the same for a disc in  $M_1$ , but for the reversed flow, one can show that the images of the discs will intersect transversely. The orbit of this intersection point gives the solution. In the non-linear case, the  $\lambda$ -lemma is used to achieve the same result. These ideas are discussed by [BD] for non-linear  $n$ -dimensional systems. We will generalize this simple geometric picture to partial differential equations. For lack of a similar lemma in our case, our method of proof will be slightly different. The main tools will be evolution operators and exponential dichotomies, which in the ordinary differential equation case generalize the above picture to non-autonomous systems. (We refer to Coppel [Co] for a discussion of these ideas in the finite dimensional setting.) We will solve a linear boundary value problem arising from (1.1) and we will use the dichotomies to construct an integral operator whose fixed point is a particular solution of the non-linear equation. Finally, using these constructions we will define an operator whose fixed point is the desired solution of the boundary value problem. For partial differential equations, different geometric ideas similar in flavor have been used for parabolic equations. D. Henry [H] offers a survey of

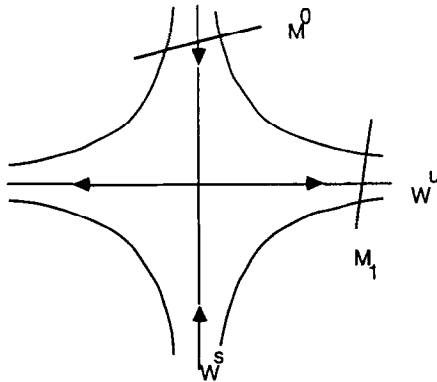


FIGURE 1

these techniques as well as a generalization of exponential dichotomies for time independent operators  $A$ .

In the finite dimensional case, that is, when  $A(t)$  is just a matrix, the systems we will be discussing often arise in the singular perturbation setting. (References on these problems can be found in Chang and Howes, [C-H] or in Smith [S]). Various systems of partial differential equations on the torus can also be treated by our methods. A simple example where our theorem could be directly applied is of the form of the Hodgekin-Huxley equation (see [H])

$$u_t = (cA + \lambda)u + H_1(t, x, u, v), \quad (1.3)(i)$$

$$v_t = B(t)v + H_2(t, x, u, v), \quad (1.3)(ii)$$

where  $v \in \mathbb{R}^2$  and where  $B(t)$  is a matrix. If the matrix  $B$  possesses an exponential dichotomy, the theorem can be used to analyse and continue solutions near a (hyperbolic) fixed point of this equation. In this paper, we will, however, show how to apply our methods to a singularly perturbed wave equation. This equation arises as the continuum limit of a chain of pendula, coupled by torsion springs. Such chains of pendula can be used to model chains of Josephson junctions, as described by M. Levi [L]. The limit we are considering is a thermodynamical limit. That is, we increase the number of pendula without scaling the coupling between the springs. We obtain an equation of the form

$$\varepsilon^2(v(t)u)_{tt} = u_{xx} + \gamma(t)u_x + \mu(t)u + f(t, x) + \varepsilon h(t, x, u), \quad (1.4)(i)$$

$$u_t(0, x) = u_t(1, x) = 0, \quad u(t, x+1) = u(t, x), \quad (1.4)(ii)$$

where  $f$  and  $h$  are periodic in  $x$  of period 1 and where  $t \in [0, 1]$ . Note that in order to write (1.4) as an evolution equation,  $x$  is the time parameter for the pendula and  $t$  refers to their spacial distribution. In this case, the results we obtain show that for small  $\varepsilon$ , the solution of (1.4) is almost independent of the boundary conditions. That is, for most of the interval, the solution of (1.4) will behave like the unique periodic solution of the equation for  $\varepsilon = 0$ . We can generalise these ideas to the case of singularly perturbed problems, that is, when the eigenvalues of  $A(t)$  have a very large modulus. One also sees that the solution does not depend strongly upon the boundary conditions.

Several criteria for existence and uniqueness of solutions of boundary problems arising from (1.1) are already known (see Pazy [Pa] or [H]). They require some spectral properties of  $A(t)$  as well as strict conditions on the non-linearity. In the time independent case, we will require the restriction of  $A$  to its eigenspaces to be hyperbolic with both stable and

unstable directions. The non-linearity will then have to be small enough and the boundary conditions will have to satisfy transversality conditions. When  $A$  is time dependent, the conditions are similar but somewhat more complicated. These conditions have the advantage of allowing us to treat systems of partial differential equations, and they can be easily checked, but they are rather restrictive on  $A$ .

In the next section, we present our main results which will be applied to (1.4) in Section 3. In the beginning of Section 3 we will also discuss finite dimensional dichotomies. The theorems will be proved in Section 4.

## 2. MAIN RESULTS

Before we can state our results, we will need some definitions. We will let  $X$  be a Hilbert space of  $C^n$ -valued functions over  $T^m$  and denote the inner product on  $X$  by  $\langle \cdot, \cdot \rangle$  with the corresponding norm  $\|\cdot\|$ . As will be seen, a typical case has  $X = H^1(T^m)$ . Define

$$\begin{aligned} A(t) &\in \text{Diff}([t_0, t_1], X) \\ &= \{\text{differential operators on } X \text{ with } C^1 \text{ time dependent coefficients}\}. \end{aligned} \quad (2.1)$$

In full analogy with the finite dimensional case we can now define exponential dichotomies.

**DEFINITION 2.1.** Let  $A(t) \in \text{Diff}([t_0, t_1], X)$  be defined on a dense domain  $\mathcal{D}(A)$ , independent of  $t$ . We say that  $A$  possesses an exponential dichotomy with projection  $P(t)$  and exponent  $\gamma$  if the following holds: There exists a projection operator  $P(t)$  defined on  $\mathcal{D}(A)$  and an evolution operator  $W(t, s)$  such that

- (i)  $W(t, s) : \mathcal{D}(A) \rightarrow \mathcal{D}(A)$  for  $t \geq s$  and for  $s \geq t$ ,  $\text{Ker}((I - P(s)) \rightarrow \mathcal{D}(A)$ ;
- (ii)  $W(t, s) P(s) = P(t) W(t, s)$ ;
- (iii)  $\|W(t, s) P(s)\| \leq K e^{-\gamma(t-s)}$ ,  $t \geq s$ ;
- (iv)  $\|W(t, s)((I - P(s)))\| \leq K e^{-\gamma(t-s)}$ ,  $s \geq t$ ;
- (v)  $\text{Ker } P(t)$  and  $\text{Ker}((I - P(t)))$  are uniformly transverse. That is, for every  $u \in \text{Ker } P(t)$ ,  $v \in \text{Ker}((I - P(t)))$ , with  $\|u\| = \|v\| = 1$ ,

$$\text{Angle}_X(u, v) > \delta$$

for some constant  $\delta$ .

We recall that a linear operator  $W(t, s)$  is said to be an evolution operator for  $A(t)$  if the following holds:

- (i)  $W(t, s) : \mathcal{D}(A) \rightarrow \mathcal{D}(A)$  for  $t \geq s$ ;
- (ii)  $W(t, t) = I$ ,  $W(t, s) = W(t, r)W(r, s) \forall r \in [t_0, t_1]$ ;
- (iii)  $W(t, s)$  is continuous;
- (iv)  $(\partial/\partial t) W(t, s) = A(t) W(t, s)$ ,  $t \geq s$ ;
- (v)  $(\partial/\partial s) W(t, s) = -W(t, s) A(s)$ ,  $t \geq s$ .

In the finite dimensional case, Definition 2.1(ii), (iii), and (iv) are the traditional definition of exponential dichotomies (see Copell [Co]). In the Hilbert space setting, one is forced to assume that the evolution operator  $W(t, s)$  is defined on  $\text{Ker}((I - P(s)))$  when  $t \geq s$  as can easily be seen from Example (1.3) in the preceding section. Since our proofs are essentially based on transversality arguments, one is, in addition, forced to assume (v). For the existence of exponential bounds as in (iii) and (iv) several criteria exist. We refer the reader to [H]. If we assume that  $e^{i\alpha \cdot x}$ ,  $\alpha \in \mathbb{Z}^m$ , are eigenvectors of  $A(t)$  and form an orthogonal basis of  $X$ , as is the case when  $X = H^l(T^m)$  or a product thereof, we have the following simple criteria for the existence of an exponential dichotomy.

**PROPOSITION 2.2.** *Let  $A(t)$  be as above.  $A(t)$  possesses an exponential dichotomy with projection  $P$  and exponent  $\gamma$  if and only if for any  $\alpha \in \mathbb{Z}^m$ , the restriction  $A_\alpha$  of  $A$  to  $\mathcal{E}_\alpha = \text{span}\{e^{i\alpha \cdot x}\}$  possesses an exponential dichotomy in the classical sense with exponent  $\gamma_\alpha \geq \gamma > 0$  and such that (v) holds uniformly in  $\alpha$ .*

Criteria for the existence of exponential dichotomies for finite dimensional systems can be found in [Co]. (See also Lemmas 3.2 and 3.3.) The details will be shown in Section 4. We can now discuss the boundary conditions for (1.1). We assume that  $A$  possesses an exponential dichotomy with projection  $P$ . We will consider sets

$$M_i = \xi_i + B_i, \quad \xi_i \in X, \quad i = 0, 1, \quad (2.2)(i)$$

where

$$B_0 \text{ is a closed subspace uniformly transverse to } \text{Ker}((I - P(t_0))) \quad (2.2)(ii)$$

and where

$$B_1 \text{ is a closed subspace uniformly transverse to } \text{Ker } P(t_1). \quad (2.2)(iii)$$

The uniform transversality is to be understood in the same sense as in

Definition 2.1(v). For  $n=2$ , a simple example of boundary conditions which may satisfy (2.2) is

$$M_i = \{w \in X / a_i w_1 + b_i w_2 = u_i(x)\}, \quad i = 0, 1,$$

where  $u_0, u_1 \in X$  and where  $a_i, b_i \in \mathbb{C}$ . We now define the Banach space

$$\mathcal{B} = C^0([t_0, t_1] : X), \quad (2.3)$$

with the norm

$$\|\cdot\|_{\mathcal{B}} = \sup_{t \in [t_0, t_1]} \|\cdot\|.$$

We will consider functions  $h(t, x, w)$  where  $t \in [t_0, t_1]$ ,  $x \in T^m$ , and  $w \in \mathbb{C}^n$  with values in  $\mathbb{C}^n$ . For any  $w \in \mathcal{B}$ , we will assume that  $h(t, x, w(t, x)) \in \mathcal{B}$ . We now have

**THEOREM 2.3.** *Assume that  $A(t) \in \text{Diff}([t_0, t_1], X)$  possesses an exponential dichotomy with projection  $P$  and exponent  $\gamma$ . Let  $M_0, M_1$  satisfy (2.2)(i)  $\rightarrow$  (iii). Then, for any  $f \in C^1([t_0, t_1] : X)$ , there exist constants  $K_1, K_2$ , and  $K_3$  depending on  $A, M_0$ , and  $M_1$  such that if*

$$K e^{-\gamma((t_1 - t_0)/2)} \leq K_1$$

and if  $h(t, x, w)$  is as above and such that for  $u, v \in \mathcal{B}$ , with  $\|u\|_{\mathcal{B}} \leq K_2$  and  $\|v\|_{\mathcal{B}} \leq K_2$ , then

$$\|h(\cdot, \cdot, u)\|_{\mathcal{B}} \leq K_3, \quad \|h(\cdot, \cdot, u) - h(\cdot, \cdot, v)\|_{\mathcal{B}} \leq K_3;$$

then

$$\frac{\partial}{\partial t} w = A(t) w + f(t, x) + h(t, x, w),$$

$$w(t_0, x) \in M_0, \quad w(t_1, x) \in M_1,$$

has a unique solution  $w$  in  $C^0([t_0, t_1] : X) \cap C^1([t_0, t_1] : X)$ , with  $\|w\|_{\mathcal{B}} \leq K_2$  which is  $C^1$  in  $t$ . Furthermore,

$$K_2 = O(\gamma^{-1} \|f\|_{\mathcal{B}} + \max_{i=0,1} \|\xi_i\|),$$

and

$$K_3 \geq c\gamma \quad \text{as } \gamma \rightarrow \infty.$$

Note that the only restrictions imposed on the non-linearity are mapping properties. In most cases, this will represent restrictions on its  $C^l$  norm.

This bound will strongly depend on the boundary values. The bigger  $\|\xi_i\|$  are, the smaller the non-linearity. As will be seen, the exponential dichotomy provides good control over the solution and allows for sharp estimates as well as an understanding of the boundary layers. Note that since the outer solution is unique, internal layers as in [KS] cannot occur. Exponential dichotomies can also be used to estimate errors due to truncation. If we now assume as in Proposition 2.2,  $e^{i\alpha \cdot x}$ ,  $\alpha \in \mathbf{Z}^m$ , are eigenvectors of  $A(t)$  that form an orthogonal basis of  $X$ , we can define a truncation operator

$$T_N w(x) = \sum_{|\alpha| \leq N} w_\alpha e^{i\alpha \cdot x}, \quad (2.4)$$

where the sum runs over all multi-indices  $\alpha = (\alpha_1, \dots, \alpha_m)$  in  $\mathbf{Z}^m$  with  $|\alpha| = |\alpha_1| + \dots + |\alpha_m| \leq N$  and where

$$w(x) = \sum_{\alpha} w_{\alpha} e^{i\alpha \cdot x}, \quad w \in X.$$

Define  $R_N = I - T_N$ . We have

**PROPOSITION 2.4.** *Under the assumptions of Theorem 2.3, a solution  $w_N$  in  $\mathcal{B}$  of the truncated boundary value problem*

$$\frac{\partial}{\partial t} w_N = A(t) w_N + T_N f(t, x) + T_N h(t, x, w_N),$$

$$w_N(t_0, x) \in M_0, \quad w_N(t_1, x) \in M_1,$$

will be such that

$$\|w - w_N\|_{\mathcal{B}} = O(\gamma_N^{-1} \|R_N f\|_{\mathcal{B}} + \max_{i=0,1} \|\xi_i\|),$$

where  $w$  is a solution of the full boundary value problem and where  $\gamma_N$  denotes the exponent of the dichotomy of  $A$  on  $R_N \mathcal{B}$ .

For our special case, we have

**COROLLARY 2.5.** *Assume  $f(t, x)$  is  $C^2$  in  $t$ ,  $H^3(S^1)$  in  $x$ , and that  $\mu, v, \gamma$  are  $C^2$  in  $t$  and satisfy  $v(t), \gamma(t) > 0$ . Then there are constants  $\varepsilon_0, K_1$ , and  $K_2$  such that if  $\varepsilon < \varepsilon_0$  and*

$$\max_{\substack{j \leq 3 \\ j \geq 4}} \sup_{\substack{t \in [0, 1], x \in S^1 \\ \|u\| \leq K_1 \|f\|_3}} |D_x^j D_u^j h(t, x, u)| \leq K_2,$$

then, the boundary value problem (1.3) has a unique classical solution  $u$ . Furthermore, for  $t \in [0, 1]$ ,  $u$  is within  $O(\varepsilon)$  of the “outer solution”

$$u_0(t, x) = \sum_k \frac{f_k(t)}{\mu(t) - k^2 + i\gamma(t)k} e^{ikx},$$

where the  $f_k$ 's are the Fourier coefficients of  $f(t, x)$ .

As will be seen, this is an easy corollary of Theorem 2.3. If one writes (1.4) as an evolution equation, the resulting operator will have an exponential dichotomy in the sense of Definition 2.2.

We will see that even though we are not using a matching argument for the existence of the solution of (1.4), one can nevertheless obtain the outer expansion in a very natural manner. Note that for small  $\varepsilon$ , the corollary implies that the dependence of the solution of (1.4) from the boundary conditions is negligible. As will become clear in the proof of this corollary, for singularly perturbed equations, the estimates of Theorem 2.4 give explicit bounds on how small  $\varepsilon$  should be (e.g., in Corollary 2.5 how small  $\varepsilon_0$  is).

We finish this section by noting that the main theorem should remain valid when  $T^m$  is replaced by any smooth compact manifold  $M$ . Proposition 2.2 would then have to be restated in terms of eigenvectors of  $A$ . That is, one would have to assume that  $A$  has a complete set of orthogonal time independent eigenvectors. However, in the case of arbitrary manifolds, several technical complications do arise and this goes beyond the scope of this paper.

### 3. A SPECIFIC EXAMPLE

#### 3.1. Finite Dimensional Dichotomies

Since we wish to apply the theorems of the previous section to (1.4), we will need some criteria for finite dimensional dichotomies. Definition 2.2 can certainly be applied to finite dimensional systems but since we will refer to Coppel's book [Co], we will give his alternative definition.

**DEFINITION 3.1.** Let  $A(t)$  be an  $n \times n$  matrix with time dependent coefficients.  $A$  is said to possess an exponential dichotomy if there exists a fundamental matrix  $X(t)$  for the system  $(d/dt)w = A(t)w$  with  $X(0) = I$  and a projection  $Q$  s.t.

- (i)  $|X(t)QX^{-1}(s)| \leq Ke^{-\gamma(t-s)}, t \geq s;$
- (ii)  $|X(t)(I-Q)X^{-1}(s)| \leq Ke^{-\gamma(s-t)}, s \geq t.$



This definition is equivalent to Definition 1.2 in the finite dimensional setting. This can be shown by setting

$$\begin{aligned} W(t, s) &= X(t) X^{-1}(s), \\ P(t) &= X(t) Q X^{-1}(t), \end{aligned}$$

or conversely

$$X(t) = W(t, 0), \quad Q = W(0, t) P(t) W(t, 0) = P(0).$$

Following Coppel, we will work on  $\mathbf{R}^n$  though the same computations can be done on  $\mathbf{C}^n$ . We have

LEMMA 3.2. *Let  $D(t)$  be a real  $2n \times 2n$  matrix in Jordan normal form. Assume that for each  $t$ ,  $D$  has no eigenvalues with zero real part and let*

$$\gamma = \min_{t \in [t_0, t_1]} \{ \operatorname{Re} \lambda / \lambda \text{ eigenvalue of } D \}.$$

*Then  $D$  possesses an exponential dichotomy with exponent  $\gamma$  and projection  $P$  on the stable eigenspaces of  $D$ .*

LEMMA 3.3. *Assume  $A$  possesses an exponential dichotomy with projection  $P$ , constant  $K$ , and exponent  $\gamma$ . If  $B$  is an  $n \times n$  matrix such that*

$$e = \sup_{t \in [t_0, t_1]} |B(t)| \leq \frac{\gamma}{4K^2},$$

*then  $A + B$  possesses an exponential dichotomy with projection  $\tilde{P}$ , exponent  $\tilde{\gamma} = \frac{1}{2}\gamma - 2K_e$ , and constant  $\tilde{K} = \frac{5}{2}K$ . Furthermore*

$$|\tilde{P}(t) - P(t)| \leq 8\gamma^{-1}K^3e \quad \forall t \in [t_0, t_1].$$

For both proofs, we refer to [Co]. Note that the second lemma is stated in [Co] (Roughness of Exponential Dichotomies) for the time interval  $[0, \infty)$  but this can easily be modified for the interval  $[t_0, t_1]$ . With these preliminaries, we can proceed to the proof of Corollary 2.5.

### 3.2. Proof of Corollary 2.5

We begin by choosing the Hilbert space  $X$  as

$$X = H^3(S^1) \times H^2(S^1) \tag{3.1}$$

with the norm

$$\left\| \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\|_\varepsilon^2 = \varepsilon^{-2} \|w_1\|_{H^3(S^1)}^2 + \|w_2\|_{H^2(S^1)}^2. \tag{3.2}$$

By our assumptions, we can expand  $f$  in Fourier series:

$$f(t, x) = \sum_k f_k(t) e^{ikx}. \quad (3.3)$$

Let

$$u_0(t, x) = \sum_k \frac{f_k(t)}{\mu(t) - k^2 + i\gamma(t)} e^{ikx}. \quad (3.4)$$

Since  $\mu, \gamma, v$ , and  $f$  are  $C^2$  in  $t$  and since  $f$  lies in  $H^3(S^1)$ ,  $u_0(t, x)$ ,  $u_{0,t}(t, x)$ , and  $u_{0,tt}(t, x)$  lie in  $H^3(S^1)$ . We can therefore change variables in our Banach space  $\mathcal{B}$  defined as in (2.3) to be  $C^0([0, 1] : X)$  (i.e., the  $u$ 's will have to be  $C^1$  in  $t$ ), by setting

$$w(t, x) = \begin{pmatrix} w_1(t, x) \\ w_2(t, x) \end{pmatrix} = \begin{pmatrix} v(t)(u(t, x) - u_0(t, x)) \\ [v(t)(u(t, x) - u_0(t, x))]_t \end{pmatrix}. \quad (3.5)$$

With this change of coordinates, (1.4) is changed to:

$$\begin{aligned} \frac{\partial}{\partial t} w(t, x) = & \begin{bmatrix} 0 & 1 \\ v^{-1}(t)/\varepsilon^2 (\Delta + \gamma(t) \nabla + \mu(t)) & 0 \end{bmatrix} w(t, x) \\ & + \tilde{f}(t, x) + \frac{1}{\varepsilon} \tilde{h}(t, x, w), \end{aligned} \quad (3.6)$$

$$\text{where } \tilde{f}(t, x) = \begin{pmatrix} 0 \\ (-vu)_{tt} \end{pmatrix} \text{ and } \tilde{h}(t, x, w) = \begin{pmatrix} 0 \\ h(t, x, v^{-1}w + u_0) \end{pmatrix}.$$

The boundary conditions are now given by

$$\begin{aligned} w_2(0, x) - \frac{v_t(0)}{v(0)} w_1(0, x) &= -v(0) u_{0,t}(0, x), \\ w_2(1, x) - \frac{v_t(1)}{v(1)} w_1(1, x) &= -v(1) u_{0,t}(1, x). \end{aligned} \quad (3.7)$$

We now verify the assumptions of Theorem 2.3. Restricting the operator  $A(t)$  defined by (3.5) to  $E_k = \text{span}\{e^{ikx}\}$ , we obtain

$$A_k(t) = \begin{bmatrix} 0 & 1 \\ v^{-1}(t)/\varepsilon^2 (\mu(t) - k^2 + i\gamma(t)k) & 0 \end{bmatrix}. \quad (3.8)$$

Note that on the  $E_k$  space, the induced norm is

$$\|(w_1, w_2)\|_{\varepsilon, k}^2 = \varepsilon^{-2} k^2 |w_1|^2 + |w_2|^2. \quad (3.9)$$

The eigenvalues of  $A_k(t)$  are the square roots of  $(v^{-1}(t)/\varepsilon^2)(\mu(t) - k^2 + i\gamma(t)k)$  which we call  $\lambda_1, \lambda_2$ . Note that  $\lambda_1, \lambda_2 \sim O(k/\varepsilon)$  but that their real part is of order  $O(1/\varepsilon)$ . Using the matrix

$$R_k(t) = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \quad (3.10)$$

we can diagonalize  $A_k(t)$ , that is

$$R_k^{-1}(t) A_k(t) R_k(t) = D_k(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \quad (3.11)$$

Introduce the new coordinates by

$$w_k = R_k(t) z_k. \quad (3.12)$$

In these coordinates, we obtain the ordinary differential equation

$$\frac{d}{dt} z_k(t) = D_k(t) z_k(t) + C_k(t) z_k(t), \quad (3.13)$$

where  $C_k(t) = R_k^{-1}(t) \dot{R}_k(t)$ . Note that this matrix has norm  $O(1)$  in the standard  $\mathbf{C}^2$  but not in the induced  $E_k$  norm. Applying Coppel's lemma of the previous paragraph, we see that  $D_k(t) + C_k(t)$  possesses an exponential dichotomy with exponent  $\gamma_k \sim O(1/\varepsilon)$ . That is, we have an evolution operator  $W_k(t, s)$  for (3.13) and a projection  $P_k(t)$  satisfying

$$\begin{aligned} W_k(t, s) P_k(s) &= P_k(t) W_k(t, s), & t, s \in [0, 1], \\ \|W_k(t, s) P_k(s)\| &\leq K e^{-\gamma_k(t-s)}, & t \geq s, \\ \|W_k(t, s)(I - P_k(s))\| &\leq K e^{-\gamma_k(t-s)}, & s \geq t. \end{aligned} \quad (3.14)$$

The norm in (3.14) is the standard  $\mathbf{C}^2$  norm for operators. In our original coordinates  $w_k$  the evolution operator  $U_k(t, s)$  is given by

$$U_k(t, s) = R_k(t) W_k(t, s) R_k^{-1}(s) \quad (3.15)$$

and the corresponding projection is

$$Q_k(t) = R_k(t) P_k(t) R_k^{-1}(t). \quad (3.16)$$

To show the existence of an exponential dichotomy for the induced  $E_k$  norm defined by (3.2), we have to estimate, for  $t \geq s$

$$\|R_k(t) W_k(t, s) P_k(s) R_k^{-1}(s)\|_{E_k}. \quad (3.17)$$

Set

$$z = R_k^{-1}(s) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = (\lambda_1 - \lambda_2)^{-1} \begin{pmatrix} -\lambda_1 w_1 - w_2 \\ -\lambda_2 w_1 + w_2 \end{pmatrix}. \quad (3.18)$$

Using the definitions of the  $\lambda$ 's and of the norm, one easily obtains that

$$\|z\| \leq C \|w\|_{\varepsilon, k}$$

and thus that the first component of  $U_k(t, s) Q_k(s) w$  can be estimated by

$$\frac{1}{2|\lambda_1|^2} K^2 e^{-2\gamma_k(t-s)} C^2 \|w\|_{\varepsilon, k}^2$$

and similarly that the second component of  $U_k(t, s) Q_k(s) w$  is estimated by

$$K^2 e^{-2\gamma_k(t-s)} C^2 \|w\|_{\varepsilon, k}^2.$$

Summarizing these last two estimates, one obtains that

$$\|U_k(y, s) Q_k(s) w\|_{\varepsilon, k} \leq K e^{-\gamma_k(t-s)} \|w\|_{\varepsilon, k}.$$

The corresponding estimate for  $U_k(t, s)(I - Q_k(s))$ , for  $s > t$ , is obtained in a similar way and we see that (3.6) possesses an exponential dichotomy on each  $E_k$ . It is now easy to see that the angle between the two kernels is bounded away from zero uniformly in both  $k$  and  $\varepsilon$ . (In the natural  $C^2$  norm, the angle is of the order of  $\varepsilon k^{-1}$ .) Thus, applying Proposition 2.2, we have shown the existence of an exponential dichotomy for Eq. (3.6). To verify that the boundary conditions given by (3.7) satisfy (2.2), one first checks that since  $u_{0,i}(t, x) \in X$  for every  $t$ , (2.2)(i) is satisfied. The transversality conditions can be explicitly verified on each  $E_k$ . Note that in the case at hand, the conditions on the non-linearity  $h$  are exactly equivalent to the ones given in Theorem 2.3 (see Palais [Pa]). We can thus apply Theorem 2.3 to (3.6) to obtain the existence and uniqueness of the solution. Now, the solution  $w$  will be  $C^1$  in  $t$ , that is, the first component has to be  $C^2$  in  $t$ . Furthermore, the first component  $w_1$  is in  $H^3(S^1)$ , that is,  $w_1(t, x)$  as well as its first two  $x$ -derivatives are continuous.

We now prove the existence and unicity of the zeroth order outer expansion.

**PROPOSITION 3.4.** *Let  $f, h, \mu, v$ , and  $\gamma$  satisfy the assumptions of Corollary 2.5 and let  $u(t, x)$  be the solution of (1.3). Let  $u_0(t, x)$  be defined as in Corollary 2.5. Then*

$$\sup_{t, x} |u(t, x) - u_0(t, x)| = O(\varepsilon).$$

Note that  $u_0$  is uniformly close to  $u$  in the  $C^0$  topology. This will be false in the  $C^1$  topology. If we did not have the non-linear term in (1.4), we could reduce (1.4) to simple ODE's by restricting ourselves to  $E_k$ . We could then find precise asymptotic expansions for  $u$ , for example by using Xiao Biao Lin's work [XBL].

As it stands, we will restrict ourselves to the zeroth order outer solution of (1.4). This provides good information on  $u$ . One could find and prove the validity of an asymptotic expansion for  $u$  but this goes beyond the scope of this paper.

*Proof.* We know that the solution  $w$  of (3.6) is such that

$$\|w\|_{\varepsilon, k} < K$$

for some constant  $K$ . Hence, its first component,  $w_1 = u - u_0$  is such that

$$\sup_{t, x} |u(t, x) - u_0(t, x)| < K\varepsilon$$

which concludes the proof of Corollary 2.5. ■

#### 4. PROOF OF THE MAIN THEOREMS

##### 4.1. Proof of Proposition 2.2

Assume that for each  $\alpha \in \mathbf{Z}^m$ , the restriction  $A_\alpha(t)$  of  $A(t)$  to  $E_\alpha = \text{span}\{e^{i\alpha \cdot x}\}$  possesses an exponential dichotomy, that is, a fundamental matrix  $W_\alpha(t, s)$  and a projection  $P_\alpha(t)$  s.t.

$$\begin{aligned} \|W_\alpha(t, s)P_\alpha(t)\|_\alpha &\leq Ke^{-\gamma_\alpha(t-s)}, & t \geq s, \\ \|W_\alpha(t, s)((I - P_\alpha(t)))\|_\alpha &\leq Ke^{-\gamma_\alpha(t-s)}, & s \geq t, \end{aligned} \quad (4.1)$$

where  $\|\cdot\|_\alpha$  denotes the induced norm on  $E_\alpha$ . The set  $\mathcal{E} = \{e^{i\alpha \cdot x} / \alpha \in \mathbf{Z}^n\}$  forms a basis of  $X$ . Hence, for any  $w \in \mathcal{B}$ , defined by (2.3),

$$w(t, x) = \sum_{\alpha \in \mathbf{Z}^m} w_\alpha(t) e^{i\alpha \cdot x}, \quad w_\alpha(t) = \begin{pmatrix} w_\alpha^{(1)}(t) \\ \vdots \\ w_\alpha^{(n)}(t) \end{pmatrix}. \quad (4.2)$$

For such a  $w$  we could formally define

$$W(t, s)w(s, x) = \sum_{\alpha \in \mathbf{Z}^m} W_\alpha(t, s)w_\alpha(s) e^{i\alpha \cdot x}, \quad (4.3)$$

and

$$P(t)w(t, x) = \sum_{\alpha \in \mathbf{Z}^m} P_\alpha(t)w_\alpha(t) e^{i\alpha \cdot x}. \quad (4.4)$$

These need not be bounded operators. (We have seen in the previous section that there are cases where

$$|W_\alpha(t, s)| \geq \tilde{K} e^{\gamma|x|(t-s)}, \quad t \geq s. \quad (4.5)$$

We therefore define  $W(t, s)$  by (4.3) on a set of finite linear combinations of elements of  $E_\alpha$ . Now let  $w \in \mathcal{B}$ . Then, for any  $N \in \mathbb{N}$  and  $t, s$  fixed,

$$\begin{aligned} & \left\| \sum_{|x| < N} W_\alpha(t, s) P_\alpha(s) w_\alpha(s) e^{i\alpha \cdot x} \right\|^2 \\ &= \sum_{i=1}^n \left\| \sum_{|x| < N} (W_\alpha(t, s) P_\alpha(s) w_\alpha(s) e^{i\alpha \cdot x})_i \right\|^2 \\ &\leq \sum_{|x| < N} \|W_\alpha(t, s) P_\alpha(s) w_\alpha(s) e^{i\alpha \cdot x}\|_\alpha^2 \\ &\leq K^2 e^{-2\gamma(t-s)} \|w(s, \cdot)\|^2. \end{aligned}$$

We let  $N$  tend to infinity to infer

$$\|W(t, s) P(s) w(s, x)\| \leq K e^{-\gamma(t-s)} \|w(s, \cdot)\|^2. \quad (4.6)$$

The proof of the other inequality is similar and will therefore be omitted. The commutation relation follows from the ones for  $W_\alpha$  and  $P_\alpha$  and the uniform transversality of the kernels of projections by assumption. ■

#### 4.2. Proof of Theorem 2.3

Let

$$H(t, x, w) = f(t, x) + h(t, x, w). \quad (4.7)$$

We now define a “variation of constants” operator:

DEFINITION 4.1. For  $w \in \mathcal{B}$ ,  $t \in [t_0, t_1]$ , define

$$\begin{aligned} K_H(w) &= \int_{t_0}^t W(t, \tau) P(\tau) H(\tau, x, w(\tau, x)) d\tau \\ &\quad - \int_t^{t_1} W(t, \tau) ((I - P(\tau)) H(\tau, x, w(\tau, x))) d\tau. \end{aligned}$$

The proof of the theorem is based on the following idea: Given the pair  $M_0, M_1$  of boundary conditions, we define

$$T: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B},$$

$$\begin{pmatrix} \tilde{w} \\ w \end{pmatrix} \rightarrow \begin{pmatrix} {}_1\tilde{w} \\ {}_1w \end{pmatrix},$$

where

$$\begin{aligned} {}_1\tilde{w} &= K_H(\tilde{w} + w), \\ {}_1w &\text{ is a solution of the boundary value problem} \end{aligned} \quad (4.8)$$

$$\begin{aligned} \frac{d}{dt} {}_1w &= A(t) {}_1w & {}_1w(t_0, x) &\in M_0 - {}_1\tilde{w}(t_0, x) \\ & & {}_1w(t_1, x) &\in M_1 - {}_1\tilde{w}(t_1, x). \end{aligned}$$

At this point, the existence of such a solution  ${}_1w$  is purely hypothetical. It will be proved in proposition 4.5.

Note that by Definition 2.1

$${}_1\tilde{w}(t_0, x) \in \text{Ker } P(t_0), \quad {}_1\tilde{w}(t_1, x) \in \text{Ker}((I - P(t_1))). \quad (4.9)$$

Assume for now that  $T$  is well defined. The next lemma relates the fixed points of  $T$  to solutions of the boundary value problem.

**LEMMA 4.2.** *Let  $(\tilde{w})$  be a fixed point of  $T$ . Then  $\tilde{w} + w$  is a solution of the boundary value problem of Theorem 2.3.*

*Proof.* Let  $u = \tilde{w} + w$ . By the definition of  $W(t, s)$  and of  $T$ , we have

$$\frac{\partial}{\partial t} u = \frac{\partial}{\partial t} \tilde{w} + \frac{\partial}{\partial t} w = H(t, x, u(t, x)) + A(t) K_H(u) + A(t) w.$$

Hence  $(\partial/\partial t) u = A(t) u + H(t, x, u(t, x))$ . ■

The proof of both theorems is therefore reduced to showing that  $T$  is well defined and is a contraction. This will be proved in the subsequent propositions. We will proceed as follows. First, we show that the boundary value problem (4.8) has a unique solution and provide estimates of the norm of the solution in terms of the boundary values. We will then proceed to show that  $T$  is a contraction on a certain ball in  $\mathcal{B} \times \mathcal{B}$ . We have

**PROPOSITION 4.3.** *Let  $u(t, x) \in \mathcal{B}$  and such that  $u(t_0, x) \in \text{Ker } P(t_0)$ ,  $u(t_1, x) \in \text{Ker}((I - P(t_1)))$ . Let  $A$ ,  $M_0$ , and  $M_1$  be as in Theorem 2.4 and let  $\gamma$  be the exponent of the dichotomy of  $A$ . Then there exist constants  $\Lambda$  and  $\mathcal{K}$  depending on  $M_0$ ,  $M_1$ , and  $P(t)$  but not on  $\gamma$  or  $u$  such that if*

$$Ke^{-\gamma((t_1 - t_0)/2)} \leq \Lambda$$

*then the boundary value problem*

$$\begin{aligned} \frac{\partial}{\partial t} w &= A(t) w, & w(t_0, x) &\in M_0 + u(t_0, x) \\ & & w(t_1, x) &\in M_1 + u(t_1, x) \end{aligned}$$

has a unique solution  $w(t, x) \in \mathcal{B}$ ,  $C^1$  in  $t$  satisfying

$$\|w\|_{\mathcal{B}} \leq \mathcal{H} \left( \max_{i=0,1} \|\xi_i + u\|_{\mathcal{B}} \right).$$

*Proof.* The idea of the proof is simple. One almost exactly mimics the existence proof for the finite linear Sil'nikov problem, that is, the existence part of the proof has essentially the same flavor as in [BD], which is only for time independent coefficients. The main part of the work consists in estimating the “size” of the solution in terms of its boundary values. This will then be needed for the contraction argument.

For simplicity, let us denote

$$\begin{aligned} u(t_0, x) + M_0 &= M_0(u), \\ u(t_1, x) + M_1 &= M_1(u). \end{aligned} \tag{4.10}$$

First note that since  $M_0(u)$  is a translate of  $M_0$  by  $u(t_0)$ ,  $M_0(u)$  is transverse to  $\text{Ker}((I - P(t_0)))$  and that similarly  $M_1(u)$  is transverse to  $\text{Ker } P(t_0)$ . We now let  $D_i$ ,  $i=0, 1$ , be a disc of radius  $r$  in  $M_i(u)$ . We denote

$$D_s = P(t_0) D_0, \quad D_u = ((I - P(t_1)) D_1). \tag{4.11}$$

For any  $\eta_s \in D_s$  and  $\eta_u \in D_u$ , we have

$$\|W(t, t_0) \eta_u\| = \|W(t, t_0) P(t_0) \eta_s\| \leq K e^{-\gamma(t-t_0)} \|\eta_s\|, \tag{4.12}(i)$$

$$\|W(t, t_1) \eta_u\| = \|W(t, t_0)((I - P(t_1)) \eta_u)\| \leq K e^{-\gamma(t_1-t)} \|\eta_u\|. \tag{4.12}(ii)$$

By uniform transversality, there exists a constant  $C$  such that for any

$$\eta_{u,s} \in D_{u,s}, \quad \|\eta_{u,s}\| \leq \|\eta_{0,1}\| + Cr. \tag{4.13}$$

Let  $\{\eta_0\} = D_s \cap D_0$  and  $\{\eta_1\} = D_u \cap D_1$ . Then, by the uniform transversality, there is a constant  $\Omega$  such that if  $\eta \in D_0$ ,  $\|\eta - \eta_0\| = r$  then

$$\|(I - P(t_0))(\eta - \eta_0)\| \geq \Omega r \tag{4.14}(i)$$

and for  $\eta \in D_1$ ,  $\|\eta - \eta_1\| = r$  then

$$\|P(t_1)(\eta - \eta_0)\| \geq \Omega r. \tag{4.14}(ii)$$

Let  $t_{1/2} = (t_1 - t_0)/2$  and define the set

$$\begin{aligned} I = \{ & w \in \mathcal{B} / P(t_{1/2}) w \in W(t_{1/2}, t_0) D_s, \\ & ((I - P(t_{1/2})) w \in W(t_{1/2}, t_1) Du \}. \end{aligned} \tag{4.15}$$



We now proceed to show that

$$((I - P(t_{1/2})): W(t_{1/2}, t_0) D_0 \cap I \rightarrow W(t_{1/2}, t_1) D_u \quad (4.16)$$

is one to one and onto. For any  $w$ , we have

$$\| W(t, t_0) P(t_0) w \| \leq K e^{-\gamma(t-t_0)} \| w \|, \quad (4.17)(i)$$

$$\| W(t, t_0)((I - P(t_0)) w) \| \geq K^{-1} e^{-\gamma(t-t_0)} \| ((I - P(t_0)) w) \|. \quad (4.17)(ii)$$

We now apply (4.12) and (4.17)(ii) to  $w = \eta - \eta_0$  for  $\eta \in \partial D_0$  to see that

$$\| W(t_{1/2}, t_0)((I - P(t_0)) \eta) \| \geq K^{-1} e^{\gamma t_{1/2}} \Omega r. \quad (4.18)$$

Combining (4.13) and (4.18), one sees that the map defined in (4.14) will be onto provided that

$$K^2 e^{-2\gamma t_{1/2}} (\| \eta_1 \| + Cr) \leq \Omega r. \quad (4.19)$$

To see that the map is one to one, note that (4.17) holds for any vector  $v$  in the tangent space to  $D_0$ . We now proceed to prove the existence and unicity of the solution of the linear boundary value problem. Let

$$S = (W(t_{1/2}, t_0) D_0) \cap I. \quad (4.20)$$

By the same argument as before,

$$\text{dist}(W(t_1, t_{1/2}) S, \text{Ker } P(t_1)) \leq K e^{-2\gamma t_{1/2}} (Cr + \| \eta_0 \|), \quad (4.21)$$

where  $C$  is as in (4.13). This, together with (4.17)(ii) shows that if

$$K e^{-2\gamma t_{1/2}} (Cr + \| \eta_0 \|) \leq \Omega r, \quad (4.22)$$

then,  $W(t_1, t_{1/2}) S \cap D_1$  is non-empty. Since (4.21) holds for the tangent spaces to  $S$  and  $\text{Ker } P(t_1)$ , if

$$K e^{-\gamma t_{1/2}} \leq \chi \quad (4.23)$$

then the intersection is transverse. Note that  $\chi$  only depends on  $M_0, M_1$ , and on the angle  $\delta$  between the kernels. We now choose  $r$  to be

$$r = \begin{cases} 0 & \text{when } \| \eta_{0,1} \| \\ \max(C, \Omega) \max(\| \eta_0 \|, \| \eta_1 \|) & \text{otherwise.} \end{cases} \quad \begin{matrix} \text{are zero} \\ \end{matrix} \quad (4.24)$$

Then, we can rewrite (4.18) and (4.22) and (4.23) as

$$K e^{-\gamma t_{1/2}} \leq A, \quad (4.25)$$

where  $A$  is independent of  $u(t, x)$ . We can now provide estimates on the size of the solution. Let the solution to the boundary value problem be denoted by  $w(t, x)$ . Then,  $w(t_{1/2}, \cdot) \in I$ . Using (4.12) and (4.18), we obtain that

$$\|P(t) w(t, \cdot)\| \leq \|\eta_0\| + Cr$$

and that

$$\|((I - P(t)) w(t, \cdot))\| \leq \|\eta_1\| + Cr$$

which shows that

$$\|w\|_{\mathcal{B}} \leq \mathcal{H} \max(\|\eta_0\|_{\mathcal{B}}, \|\eta_1\|_{\mathcal{B}}). \quad (4.26)$$

Using the properties of the evolution operator, we easily find that this solution is  $C^1$  in  $t$ , which concludes the proof of the proposition. ■

We can now continue the proof of Theorem 2.4. Given  $f(t, x) \in \mathcal{B}$ , choose constants  $K_2$  and  $K_3$  so that

$$K_2 \geq \frac{K}{\gamma} \max(1, \mathcal{H}) \|f\|_{\mathcal{B}} + \mathcal{H} \max_i (\|\xi_i\|) + 1. \quad (4.27)(i)$$

For  $\|u\|_{\mathcal{B}} \leq K_2$  and  $\|v\|_{\mathcal{B}} \leq K_2$  require that

$$\begin{aligned} & \|h(\cdot, \cdot, u)\|_{\mathcal{B}}, \|h(\cdot, \cdot, w) - h(\cdot, \cdot, v)\|_{\mathcal{B}} \\ & < \max(1, \mathcal{H})^{-1} \frac{\gamma}{4K} = K_3. \end{aligned} \quad (4.27)(ii)$$

Note that when  $\gamma$  is large, this implies that  $K_3 \sim O(\gamma)$ . By integration, one easily sees that for any  $u$  and  $v$  with norm less than  $K_2$ ,

$$\|K_H(v)\|_{\mathcal{B}} \leq K_2, \quad \|K_H(u) - K_H(v)\|_{\mathcal{B}} \leq \frac{1}{2} \|u - v\|_{\mathcal{B}}. \quad (4.28)$$

Let  $\tilde{w}, w \in \mathcal{B}$ ,  $\|w\|_{\mathcal{B}}, \|\tilde{w}\|_{\mathcal{B}} \leq K_2$ , and define  $K_1$  as follows,

$$K_1 \leq A, \quad (4.30)$$

where  $A$  is defined in Proposition 4.5. Assuming that the exponential dichotomy of  $A$  satisfies

$$Ke^{-\gamma((t_1 - t_0)/2)} \leq K_1, \quad (4.31)$$

we see that  $T$  is well defined. Let  $U(K_2) = \{\|\tilde{w}\|_{\mathcal{B}} \leq K_2\} \times \{\|w\|_{\mathcal{B}} \leq K_2\}$ . For  $w$  and  $\tilde{w}$  in  $U(K_2)$ , let  $T(\tilde{w}) = \binom{\tilde{w}}{w}$ . Then by (4.27) and (4.28)

$$\|{}_1\tilde{w}\|_{\mathcal{B}} \leq K_2, \quad (4.32)(i)$$

and, using (4.26),

$$\| {}_1w \|_{\mathcal{B}} \leq K_2. \quad (4.32)(ii)$$

We now check that  $T$  is a contraction of  $U(K_2)$ . Let  $(\tilde{w})$  and  $(\tilde{u})$  be in  $U(K_2)$ . Denote their image under  $T$  by a left subscript 1. By Proposition 4.3, (4.27), and (4.28) we have

$$\| {}_1\tilde{w} - {}_1\tilde{u} \| \leq \mu(\| \tilde{w} - \tilde{u} \|_{\mathcal{B}} + \| w - u \|_B), \quad \mu < \frac{1}{2}. \quad (4.33)$$

As in Section 2, let  $\{\xi_0\} = M_0 \cap \text{Ker}((I - P(t_0)))$  and  $\{\xi_1\} = M_1 \cap \text{Ker} P(t_1)$ . By the definition of  $T$ ,  ${}_1w - {}_1u$  is a solution of the boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t} ({}_1w - {}_1u) &= A(t)({}_1w - {}_1u), \\ ({}_1w - {}_1u)(t_0, x) &\in B_0 - ({}_1\tilde{w} - {}_1\tilde{u})(t_0, x), \\ ({}_1w - {}_1u)(t_1, x) &\in B_1 - ({}_1\tilde{w} - {}_1\tilde{u})(t_1, x), \end{aligned} \quad (4.34)$$

where the  $B_i$ 's are defined in (2.2)(i). To see this, note that at  $t_0$ ,  ${}_1w(t_0)$  belongs to  $M_0 - {}_1\tilde{w}(t_0)$  and  ${}_1u(t_0)$  belongs to  $M_0 - {}_1\tilde{u}(t_0)$ . Subtracting, we see that  ${}_1w(t_0) - {}_1u(t_0)$  will belong to  $M_0 - \xi_0 - {}_1\tilde{w}(t_0) + {}_1\tilde{u}(t_0)$  and hence (4.34). Using Proposition 4.3 with  $B_0$  and  $B_1$  instead of  $M_0$  and  $M_1$ , we see that (4.25) implies that

$$\| {}_1w - {}_1u \|_{\mathcal{B}} \leq \mathcal{H} \| {}_1\tilde{w} - {}_1\tilde{u} \|_{\mathcal{B}}. \quad (4.35)$$

Again, by Proposition 4.3 and our choice of  $K_2, K_3$ , we see that

$$\| {}_1w - {}_1u \|_{\mathcal{B}} \leq \mu(\| \tilde{w} - \tilde{u} \|_{\mathcal{B}} + \| w - u \|_{\mathcal{B}}), \quad (4.36)$$

where  $\mu < \frac{1}{2}$ . But this shows that  $T$  is a contraction on  $U(K_2)$  equipped with the norm  $\|(\tilde{w})\| = \|\tilde{w}\|_{\mathcal{B}} + \|w\|_{\mathcal{B}}$ .  $T$  has therefore a fixed point in  $U(K_2)$  and by Lemma 4.2, we have a unique solution of the non-linear boundary value problem. It is now easy to see that by the definition of  $K_H$  and by Proposition 4.3, for any  $u \in \mathcal{B} \times \mathcal{B}$ ,  $Tu$  is  $C^1$  in  $t$  and thus that Theorem 2.3 is proved. ■

#### 4.3. Truncations

We now proceed to prove Proposition 2.4. We let  $T_N$  and  $R_N$  be defined as in (2.4). Note that  $T_N$  commutes with both  $A$  and  $\partial_t$ . Let  $w_N$  be a solution of the truncated boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t} w_N &= A(t) w_N + T_N f(t, x) + T_N h(t, x, w_N), \\ w_N(t_0, x) &\in T_N M_0, \quad w_N(t_1, x) \in T_N M_1. \end{aligned}$$

By Theorem 2.3, there is a unique solution of this problem in  $T_N\mathcal{B}$ . Let  $w$  be the solution of the full boundary value problem. Then

$$R_N w = w - w_N$$

satisfies

$$\frac{\partial}{\partial t} R_N w = A(t) R_N w + R_N f(t, x) + R_N h(t, x, w_N + R_N w),$$

$$R_N w(t_0, x) \in M_0, R_N w(t_1, x) \in M_1.$$

The new non-linearity

$$g(t, x, R_N w) = R_N h(t, x, w_N + R_N w) - R_N h(t, x, w_N)$$

satisfies the conditions of Theorem 2.3 on  $R_N\mathcal{B}$  and thus,

$$\|R_N w\|_{\mathcal{B}} = O(\gamma_N^{-1} \|R_N f + R_N h(\cdot, \cdot, w_N)\|_{\mathcal{B}} + \max_{i=1,2} \|R_N \xi_i\|)$$

which concludes the proof.

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